Math 206B Lecture 2 Notes

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1 Representation Theory of Finite Groups

This is meant to remind you of basic results in representation theory. Good references are Chapter 1 of Sagan's book and the book by Fulton and Harris.[1]

1.1 Conjugacy classes and characters

Let G be a finite group. There is an action of G on G in the following way: $a \cdot g = aga^{-1}$. This is the action of conjugation, where conjugacy classes are the orbits of the action. In other words, $g \sim h$ if $g = aha^{-1}$ for some $a \in G$; conjugacy classes are equivalence classes under this relation. We will denote $c(G)$ as the number of conjugacy classes of G.

Example 1.1. Let $G = \mathbb{Z}_n$ be the cyclic group of order n. Then $c(\mathbb{Z}_n) = n$.

Example 1.2. Let $G = S_n$. Then $c(S_n) = p(n)$, the number of integer partitions of n.

Definition 1.1. A **character** $\chi_1 : G \to \mathbb{C}$ is a function such that $\chi(g) = \chi(h)$ whenever $g \sim h$.

Example 1.3. The trivial character is $\chi(g) = 1$ for all g.

Example 1.4. Let $G = S_n$. The sign character is $\chi(\sigma) = \operatorname{sign}(\sigma)$.

Theorem 1.1. $\dim(\operatorname{span}(\textit{characters } \chi) = c(G)$.

Definition 1.2. The **inner product** on characters is defined as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Example 1.5. If $n \geq 2$, then

$$\langle \chi, \operatorname{sign} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot 1 = \frac{1}{n!} \left(\frac{n!}{2} - \frac{n!}{2} \right) = 0.$$

Example 1.6. Suppose $G = \mathbb{Z}_n$. Let $\omega = e^{2\pi i/n}$. For each j, we have a character $\chi_j(k) = \omega^{jk}$.

[1] This is Professor Pak's opinion. I hate this book.

1.2 Linear representaitons

Let $V = \mathbb{C}^d$. The group $\mathrm{GL}(V) = \mathrm{GL}_d(\mathbb{C})$ is the group of automorphisms of V.

Definition 1.3. A **representation** is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Here are operations we can do on representations:

1. If we have $\rho : G \to \mathrm{GL}(V), \pi : G \to \mathrm{GL}(W)$, then we can form $\rho \oplus \pi : G \to \mathrm{GL}(V \oplus W)$ by acting on the individual parts of the vector space by the respective actions.
2. We can form $\rho \otimes \pi : G \to \mathrm{GL}(V \otimes W)$. The dimension of $\rho \otimes \pi$ is $\dim(V)\dim(W)$.
3. Reduced representations: If we have $\rho : G \to \mathrm{GL}(V)$ and $H \leq G$, we can define $\rho \downarrow_H^G$ as the restriction of ρ to H.
4. Induced representations: If we have $\pi : H \to GL(W)$, there is an induced representation $\pi \uparrow_H^G : G \to \mathrm{GL}(W^{\otimes h})$, where $h := [G : H] = |G|/|H|$.

Example 1.7. The **trivial representation** maps $g \mapsto \mathrm{id}_V$ for all $g \in G$.

Example 1.8. The **regular representation** $\pi : G \to \mathrm{GL}(\mathbb{C}^{|G|})$ acts on a basis indexed by all $g \in G$ by $a \cdot v_g = v_{ag}$.

Example 1.9. The **natural representation** $\rho : S_n \to \mathrm{GL}_n(\mathbb{C})$ sends σ to its permutation matrix (applying the permutation to the basis vectors $\{e_1, \ldots, e_n\}$).

If we have a representation $\rho : G\, to\, \mathrm{GL}(V)$, we can define its character χ_rho by $\chi_\rho(g) = \mathrm{tr}([\rho(g)])$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$. So χ_ρ is in fact a character in the previous sense.

Remark 1.1. Even though $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, then $\mathrm{tr}(ABC) \neq \mathrm{tr}(CBA)$.

We also have $\mathrm{tr}(A+B) = \mathrm{tr}(A)+\mathrm{tr}(B)$. However, we do not have $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

Example 1.10. Let π be the regular representation of G. Then

$$\chi_\pi(\sigma) = \begin{cases} n! & \sigma = 1 \\ 0 & \sigma \neq 1. \end{cases}$$

Example 1.11. Let ρ be the natural representation of S_n. Then $\chi_\rho(\sigma)$ is the number of fixed points of σ.

Example 1.12. Here is an example of a reduced representation. Let $G = S_n$, and $H = \mathbb{Z}_n$. Let ρ be the natural representation of S_n. Then $\rho \downarrow_{\mathbb{Z}_n}^{S_n}$ is the regular representation of $\mathbb{Z}_n$.